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A Question on Finite Moufang Veblen–Wedderburn Systems*

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1. INTRODUCTION

In a recent paper Kallaher [3] investigates (left) Veblen–Wedderburn systems (and the corresponding projective planes) in which the Moufang identity

$$(xy)(zx) = (x(yz))\,x, \tag{1.1}$$

holds. Such a system is called a Moufang Veblen–Wedderburn system (MVW system). Fields, nearfields and Cayley–Dickson division algebras are examples of MVW systems. A proper MVW system is one in which the other distributive law does not hold. The only proper MVW systems known are the nearfields. Kallaher [3] obtains two sets of conditions under which an MVW system is a nearfield. Whether or not there exist proper MVW systems which are not nearfields is still an open question. We answer this question partially in Section 3. Incidentally we obtain in Section 2 a rather interesting result that if the multiplicative loop of a finite (left) VW system contains a cyclic subgroup A of certain order then A generates the VW system as a right vector space over the kern.

2. VEBLEN–WEDDERBURN SYSTEMS

Let F be a set consisting of at least two elements, 0 and 1, and let two binary operations, addition $(+)$ and multiplication $(\cdot)$ be defined on F. The system $F(+, \cdot)$ is called a left Veblen–Wedderburn system (left VW system) if the following conditions are satisfied:

(i) $F(+)$ is an abelian group, with identity 0;

(ii) $F'(\cdot)$ is a loop with identity 1, where $F' = F - \{0\}$;

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(iii) $x \cdot (y + z) = x \cdot y + x \cdot z$ for all x, y, z in F;

(iv) $0 \cdot x = x \cdot 0 = 0$ for all x in F;

(v) if r, s and t are from F and $r \neq s$, then there is a unique x in F such that $r \cdot x = s \cdot x + t$.

It is easy to show that (v) follows from (i)–(iv) if F is finite. Finiteness of F further implies that F is of prime power order. Throughout this paper a (left) VW system will be called simply a VW system. For convenience ab is used in place of $a \cdot b$.

DEFINITION. The set $K = \{a \in F \mid (xy)a = x(ya)$ and $(x + y)a = xa + ya$ for all x, y in $F\}$ is called the kern of the VW system $F(+, \cdot)$.

It is known that $K(+, \cdot)$ is in general an associative division ring and in particular a field if F is finite. Further, $F(+, \cdot)$ is a right vector space over its kern K (Pickert [4]).

DEFINITION. A set $T = \{x_0, x_1, \ldots, x_k\}$ with $x_j \in F'$ is said to be independent over K if and only if $\sum_{j=0}^{k} x_j a_j = 0$ with $a_j \in K$ for $j = 0, 1, \ldots, k$ implies $a_0 = a_1 = \cdots = a_k = 0$. If the set T is not independent over K it is said to be dependent over K.

In the rest of this section $F(+, \cdot)$ stands for a finite VW system of order q^d with kern K of order $q = p^u$, p is a prime, u and d are natural numbers and $d > 2$.

LEMMA 2.1. *If g is an element from the loop $F'(\cdot)$ such that*

(i) *g generates a cyclic subgroup G of order t;*

(ii) *the set $T = \{1, g, g^2, \ldots, g^k\}$ is independent over K where $k \leqslant t - 1$;*

(iii) *$T \cup \{g^{(k+1)}\}$ is dependent over K, then $T \cup \{g^{(k+i)}\}$ is dependent over K for all $i \geqslant 1$ such that $k + i \leqslant t$.*

Proof. The lemma is trivially true if $k + 1 = t$. Suppose $k + 1 < t$. Since $T \cup \{g^{(k+1)}\}$ is dependent over K we have $g^{(k+1)} = \sum_{j=0}^{k} g^j a_j$ where $a_j \in K$ for $0 \leqslant j \leqslant k$ and not all a_j are equal to zero. Further independence of T over K implies $a_0 \neq 0$. Suppose $a_0 = 0$. Then $(g^{-1})(g^{(k+1)} - \sum_{j=1}^{k} g^j a_j) = 0 = g^k - \sum_{j=1}^{k} g^{(j-1)} a_j$ which implies that T is dependent over K, a contradiction. The lemma is true for $i = 1$ by hypothesis. Suppose the lemma is true for integers $1, 2, \ldots, i - 1$. Then

$$g^{(k+i-1)} = \sum_{j=0}^{k} g^j b_j \tag{2.1}$$

where $b_j \in K$ and $b_j \neq 0$ for all j. After some simplifications using the left distributive law and properties of the kern we have

$$g^{(k+i)} = \sum_{j=0}^{k} g^j c_j \tag{2.2}$$

(by multiplying both sides of (2.1) on the left by g) where $c_0 = a_0 b_k$ and $c_j = a_j b_k + b_{j-1}$. If $b_k \neq 0$, then $c_0 \neq 0$. Suppose $b_k = 0$. Then there is an integer r such that $b_{r-1} \neq 0$ and $b_j = 0$ for $r \leqslant j \leqslant k$. Then $c_r = b_{r-1} \neq 0$. Hence the lemma.

LEMMA 2.2. *If $a_i \in K$ for $i = 0, 1, \ldots, k$ with $a_k \neq 0$ and g is a power associative element in $F'(\cdot)$ then there are at most k elements x in $G = \langle g \rangle$ satisfying*

$$\sum_{i=0}^{k} x^i a_i = 0. \tag{2.3}$$

Proof. The lemma follows trivially if $k \geqslant |G|$, the order of G. We may take $k < |G|$. Suppose there are $k+1$ distinct elements $x_1, x_2, \ldots, x_{k+1}$ in G satisfying (2.3).

Let $h(r, s, t)$ and $f(r, s)$ be functions defined by (using $m(i)$ and m_i interchangeably)

$$h(r, s, t) = \sum (x_s)^{m(0)} (x_{s+1})^{m(1)} \cdots (x_{s+t})^{m(t)} \tag{2.4}$$

where the summation extends over all partitions $[m_0, m_1, \ldots, m_t]$ of r and $s + t \leqslant k + 1$, and

$$f(r, s) = \sum_{i=r}^{k} h(i - r, s, r)\, a_i . \tag{2.5}$$

It may be noted that the above functions are unambiguously defined since the elements x_i are from an associative set (in fact a group) G and $a_i \in K$. It is claimed that the function f satisfies the following relation.

$$f(r, s) - f(r, s+1) = x_s f(r+1, s) - x_{s+r} f(r+1, s). \tag{2.6}$$

The following two relations are needed in this connection which follow easily from the definition of the function $h(r, s, t)$.

$$h(r+1, s, t) = h(1, s, 0)\, h(r, s, t) + h(r+1, s+1, t-1), \tag{2.7}$$

$$h(r+1, s, t) = h(1, s+t, 0)\, h(r, s, t) + h(r+1, s, t-1). \tag{2.8}$$

Using (2.7), (2.8), and the fact $h(0, s, t) = 1$ for all s and t we obtain

$$\begin{aligned} x_s f(r+1, s) - x_{s+r} f(r+1, s) &= \sum_{i=r+1}^{k} (x_s h(i-r-1, s, r+1) a_i) \\ &\quad - \sum_{i=r+1}^{k} (x_{r+s} h(i-r-1, s, r+1) a_i) \\ &= \sum_{i=r+1}^{k} (h(i-r, s, r+1) - h(i-r, s+1, r)) a_i \\ &\quad - \sum_{i=r+1}^{k} (h(i-r, s, r+1) - h(i-r, s, r)) a_i \\ &= \sum_{i=r}^{k} h(i-r, s, r) a_i - \sum_{i=r}^{k} h(i-r, s+1, r) a_i \\ &= f(r, s) - f(r, s+1). \end{aligned}$$

The $k+1$ equations satisfied by x_j, $1 \leqslant j \leqslant k+1$ may now be written as

$$f(0, s) = 0 \qquad \text{for} \qquad 0 \leqslant s \leqslant k+1. \tag{2.9}$$

It is asserted that $f(r, s) = 0$ for $0 \leqslant r \leqslant k$ and $r + s \leqslant k+1$. We induct on r. Expression (2.9) implies the truth of the assertion for $r = 0$ and $s \leqslant k+1$. Suppose $f(r, s) = 0$ for $0 \leqslant r \leqslant t$ and $s + r \leqslant k+1$. Then (2.6) implies $x_s f(t+1, s) - x_{s+r} f(t+1, s) = 0$. Since $x_s \neq x_{s+r}$ we may conclude that $f(t+1, s) = 0$. Thus $f(r, s) = 0$ for $0 \leqslant r \leqslant k$ and $s - r \leqslant k+1$. We obtain after some computations

$$f(k-1, 1) - f(k-1, 2) = 0 = (x_1 - x_{k+1})\, a_k\,. \tag{2.10}$$

Since $x_1 \neq x_{k+1}$ (2.10) implies $a_k = 0$, a contradiction. Hence the lemma.

LEMMA 2.3. *If $a_i \in K$ and $G = \langle g \rangle$, g a power associative element from the loop $F'(\cdot)$, and there exist $x_0, \ldots, x_k$ in G with $0 = \sum_{i=0}^{k} x_i a_i$, then $\sum_{i=0}^{k} x_i t a_i = 0$ where $t = q^j$ for $0 \leqslant j \leqslant d-1$.*

Proof. Once again using $m(i)$ and m_i interchangeably we first show that for all natural numbers n

$$S_n = \sum \left(\prod_{i=0}^{k} x_i^{m(i)}\right)\left(\prod_{i=0}^{k} a_i^{m(i)}\right)\left(n! \Big/ \left(\prod_{i=0}^{k} (m_i!)\right)\right) = 0, \tag{2.11}$$

where the summation extends over all partitions $[m_0, m_1, \ldots, m_k]$ of n and yr for any $y \in F$ and any natural number r is defined inductively:

$$y(1+1) = y + y, \qquad yr = y(r-1) + y, \qquad r \geqslant 2.$$

Since $S_1 = x_1a_1 + x_2a_2 + \cdots + x_ka_k = 0$, (2.11) is true for $n = 1$. Suppose (2.11) is true for natural numbers $1, 2, \ldots, n$. Then for any integer r with $0 \leqslant r \leqslant k$, $x_rS_na_r = 0$. Using the left distributive law, properties of the kern and the fact that G is an associative set we obtain after simplification

$$0 = x_rS_na_r = \sum \Big(\prod_{i=0}^{k} x_i^{n(i)}\Big)\Big(\prod_{i=0}^{k} a_i^{n(i)}\Big)\Big((n!)n_r \Big/ \Big(\prod_{i=0}^{k} (n_i!)\Big)\Big), \tag{2.12}$$

where the summation extends over all partitions $[n_0, n_1, \ldots, n_k]$ of $n+1$ with $n_r \geqslant 1$. Since $x_rS_na_r = 0$ for $0 \leqslant r \leqslant k$ we obtain

$$\sum_{r=0}^{k} x_rS_na_r = 0. \tag{2.13}$$

Let $[m_0, m_1, \ldots, m_k]$ be an arbitrary partition of $n+1$. From (2.12) it follows that no term of type $(\prod_{i=0}^{k} x_i^{m(i)})(\prod_{i=0}^{k} a_i^{m(i)})$ occurs in $x_rS_na_r$, if $m_r = 0$, and for every r for which $m_r \neq 0$, $x_rS_na_r$ contains a term of the above type. Thus the sum of all terms of the type mentioned above contained in (2.13) may be expressed as

$$\Big(\prod_{i=0}^{k} x_i^{m(i)}\Big)\Big(\prod_{i=0}^{k} a_i^{m(i)}\Big)\Big((n)! \sum m_i \Big/ \Big(\prod_{i=0}^{k} (m_i!)\Big)\Big) = T, \tag{2.14}$$

where $\sum m_i$ is the sum of all $m_i \neq 0$ obviously $\sum m_i = n+1$. Thus from (2.14) we obtain

$$T = \Big(\prod_{i=0}^{k} x_i^{m(i)}\Big)\Big(\prod_{i=0}^{k} a_i^{m(i)}\Big)\Big((n+1)! \Big/ \Big(\prod_{i=0}^{k} (m_i!)\Big)\Big). \tag{2.15}$$

(2.13) now implies in view of (2.15)

$$S_{n+1} = \sum \Big(\prod_{i=0}^{k} x_i^{m(i)}\Big)\Big(\prod_{i=0}^{k} a_i^{m(i)}\Big)\Big((n+1)! \Big/ \Big(\prod_{i=0}^{k} (m_i!)\Big)\Big) = 0, \tag{2.16}$$

where the summation extends over all partitions $[m_0, m_1, \ldots, m_k]$ of $n+1$. Hence (2.11) is true for all n.

Let $[m_0, m_1, \ldots, m_k]$ be a partition of the prime p. Then two cases arise: (i) $m_r = p$ for some r and (ii) $m_r < p$ for all r. It then easily follows that $(p!/(\prod_{i=0}^{k} (m_i!)))$ $= 1$ in the first case and is a multiple of p in the second

case. By letting $n = p$ in (2.11) and using the fact $yp = 0$ for all y in F we obtain

$$S_p = \sum_{i=0}^{k} x_i{}^p a_i{}^p = 0. \tag{2.17}$$

We now obtain the following relations:

$$\sum_{i=0}^{k} x_i{}^t a_i{}^t = 0 \qquad \text{where} \quad t = p^r \qquad \text{for} \quad 0 \leqslant r \leqslant ud - 1. \tag{2.18}$$

We induct on r. (2.18) becomes (2.17) if $r = 1$ and therefore (2.18) is true for $r = 1$. Suppose (2.18) is true for intigers $1, 2, \ldots, r$. Let $y_i = x_i{}^t$ and $b_i = a_i{}^t$ where $t = p^r$ for $0 \leqslant i \leqslant k$. Then (2.18) may be written as

$$\sum_{i=0}^{k} y_i b_i = 0, \tag{2.19}$$

where $y_i \in G$ and $b_i \in K$ for $0 \leqslant i \leqslant k$. Thus the hypothesis of the lemma is satisfied if x_i and a_i are replaced by y_i and b_i respectively for $0 \leqslant i \leqslant k$. We then obtain from (2.17)

$$\sum_{i=0}^{k} y_i{}^p b_i{}^p = \sum_{i=0}^{k} x_i{}^t a_i{}^t = 0 \qquad \text{where} \quad t = p^{r+1}. \tag{2.20}$$

Thus (2.18) is true for $0 \leqslant r \leqslant u\,d - 1$. The lemma now follows from (2.18) and the fact $a_i{}^t = a_i$ where $t = p^{uj}$ for all i and j.

Let e be a prime which divides $p^{ud} - 1$ but does not divide $p^{ut} - 1$ for $0 < t < d$. Prime e exists in all cases except when $p = 2$, $u = 1$ and $d = 6$ ([*1*] Corollary 2, p. 358).

THEOREM 2.1. *Let $F(+, \cdot)$ be a finite (left) Veblen–Wedderburn system of order q^d with kern K of order $q = p^u$ where p is a prime, u and d are natural numbers, $d > 2$ and if $p = 2$ and $u = 1$ then $d \neq 6$. Let e be a prime which divides $p^{ud} - 1$ but does not divide $p^{ut} - 1$ for $0 < t < d$. If the loop $F'(\cdot)$ contains a power associative element g of order e, then the subgroup $G = \langle g \rangle$ generates $F(+, \cdot)$ as a right vector space over K. Further the set $T = \{1, g, g, \ldots, g^{d-1}\}$* is a basis.

Proof. The theorem is proved if we show that T is an independent set over K. Since the prime e does not divide the order of K' the element g does not belong to K and consequently the set $\{1, g\}$ is independent over K. By Lemma 2.1 there is an integer k such that the set $\{1, g, g^2, \ldots, g^{k-1}\}$ is independent whereas the set $\{1, g, g^2, \ldots, g^k\}$ is dependent over K. It is claimed that $k = d$. Since $F(+, \cdot)$ is of dimension d over K we have $k \leqslant d$. Suppose

$k < d$. Then there are elements b_i in K for $0 \leqslant i \leqslant k$ with $b_k \neq 0$ and $b_i \neq 0$ for at least one $i < k$, such that $\sum_{i=0}^{k} g^i b_i = 0$. Lemma 2.3 then implies

$$\sum_{i=0}^{k} g^t b_i = 0 \qquad \text{where} \quad t = q^{ji} \qquad \text{for} \quad 0 \leqslant j < d. \tag{2.21}$$

We now claim that the elements g^v are pairwise distinct for $0 \leqslant j < d$, where $v = q^j$. Suppose there are integers r and s such that $0 \leqslant s < r$, $g^m = g^n$ where $m = q^r$, $n = q^s$, and $r, s < d - 1$. Since g is of order e we have $q^r \equiv q^s \pmod{e}$. This congruence implies that e divides $q^{r-s} - 1$ where $0 < r - s < d$, contradicting the definition of e. (2.21) now implies that the equation $\sum_{i=0}^{k} x^i b_i = 0$ has $d > k$ solutions g^m in G where $m = q^j$, $0 \leqslant j < d$, contradicting Lemma 2.2. Hence $d = k$ and the proof of the theorem is now completed.

We now state the following obvious result without proof.

LEMMA 2.4. *If a (left) VW system $F(+, \cdot)$ (finite or infinite) is a vector space of dimension two over its kern K, then the set $\{1, g\}$ generates $F(+, \cdot)$ as a right vector space over K where g is any element in F which does not belong to K.*

3. MOUFANG VEBLEN–WEDDERBURN SYSTEMS

In this section $F(+, \cdot)$ denotes a left VW system in which the multiplicative (Moufang) identity

$$(xy)(zx) = (x(yz))x, \tag{1.1}$$

holds. A (non-empty) set A of a Moufang loop G is called associative if $(ab)c = a(bc)$ for all a, b, c in A.

The following results are needed in the proofs of this section.

RESULT 1. ([2], Lemma 4.1) In a Moufang loop G the equation $a(bc) = (ab)c$ implies each of the equations obtained by permuting a, b, c or replacing any of these elements by their inverses.

RESULT 2. ([2], Corollary 4.3) Every maximal associative subloop of a Moufang loop G is a maximal associative subset of G.

RESULT 3. Let $F(+, \cdot)$ be a finite nearfield of dimension d over its kern K. Then there exists an element g in F such that the set $T = \{g^i \mid 0 \leqslant i \leqslant d - 1\}$ is a basis for $F(+, \cdot)$ over K.

Proof. If $F(+, \cdot)$ is a field the statement is clear. If $F(+, \cdot)$ is a proper nearfield with kern $\neq GF(2)$ the result follows from the Sylow theorem, Theorem 2.1, and Lemma 2.4. But there exists no proper finite nearfield with kern $GF(2)$ (Zassenhaus [5]).

Let $F(+, \cdot)$ be an MVW system with kern K. Let S be any subset of F. The vector space generated by S over K (as a right vector space) is denoted by $V(S; K)$.

LEMMA 3.1. *If S is an associative subset of an* MVW *system $F(+, \cdot)$ with kern K, then $V(S; K)$ is an associative subset of $F(+, \cdot)$.*

Proof. Let $u, v, w_i \in F$, $i = 1,..., m$, such that

$$(uv)\, w_i = u(vw_i), \qquad i = 1, 2,..., m. \tag{3.1}$$

Using (3.1), left distributive law and the properties of the kern we have, for $a_i \in K$,

$$\begin{aligned}(uv) \sum_{i=1}^{k} (w_i a_i) &= \sum_{i=1}^{k} ((uv)(w_i a_i)) = \sum_{i=1}^{k} (((uv)w_i)a_i) \\ &= \sum_{i=1}^{k} ((u(vw_i))a_i) = \sum_{i=1}^{k} (u((vw_i)a_i)) = \sum_{i=1}^{k} u((vw_i)a_i) \\ &= u \sum_{i=1}^{k} (v(w_i a_i)) = u\left(v \sum_{i=1}^{k} (w_i a_i)\right).\end{aligned}$$

Thus

$$(uv) \sum_{i=1}^{k} (w_i a_i) = u\left(v \sum_{i=1}^{k} (w_i a_i)\right). \tag{3.2}$$

Let $x, y, z \in V(S; K)$. Then $x = \sum_{i=1}^{n} x_i a_i$, $y = \sum_{i=1}^{m} y_i b_i$ and $z = \sum_{i=1}^{t} z_i c_i$ where $x_i, y_j, z_k \in S$, $a_i, b_j, c_k \in K$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ and $1 \leqslant k \leqslant t$. If one of x, y, and z is zero, then $(xy)\, z = 0 = x(yz)$. Suppose $x, y, z \in F'$. If we let $u = x_r$, $v = y_s$, $w_i = z_i$ and $a_i = c_i$ in (3.2) we obtain

$$(x_r y_s)\, z = x_r(y_s z) \quad \text{for} \quad 0 \leqslant r \leqslant n \quad \text{and} \quad 0 \leqslant s \leqslant m. \tag{3.3}$$

Result 1 and (3.3) then imply

$$(x_r z)\, y_s = x_r(z y_s) \quad \text{for} \quad 0 \leqslant r \leqslant n \quad \text{and} \quad 0 \leqslant s \leqslant m. \tag{3.4}$$

Repeating the above process twice each time using Result 1 and (3.2) we have $(xy)\, z = x(yz)$.

LEMMA 3.2. *Let $F(+, \cdot)$ be an* MVW *system with kern K. If $A(\cdot)$ is a maximal associative subloop contained in the loop $F'(\cdot)$, then $B(+, \cdot)$ is a maximal nearfield contained in $F(+, \cdot)$ where $B = A \cup \{0\}$.*

Proof. Let $D = V(A; K)$. Lemma 3.1 implies that D' is an associative subset of $F'(\cdot)$. Maximality of $A(\cdot)$ and Result 2 force $D' = A$ and consequently $D = B$. Since D is closed under addition and $B' = D' = A$ is an associative loop, we may conclude that $B(+, \cdot)$ is a nearfield. Maximality of $B(+, \cdot)$ in $F(+, \cdot)$ follows from the maximality of $A(\cdot)$ in $F'(\cdot)$.

THEOREM 3.1. *Let $F(+, \cdot)$ be an* MVW *system of dimension two over its kern K. Then $F(+, \cdot)$ is a nearfield.*

Proof. Let g be an element from F which does not belong to K. Then $F = V(T; K)$ where $T = \{1, g\}$. Let H be the subgroup generated by g (H exists since the elements are power associative). By Zorn's lemma there exists a maximal associative subloop A containing H and contained in $F'(\cdot)$. Lemma 3.2 now implies that $A \cup \{0\}$ is a nearfield. The theorem now follows from the fact $g \in A$.

THEOREM 3.2. *Every finite Moufang Veblen–Wedderburn system is a nearfield.*

Proof. The case of a finite MVW system of dimension two over its kern is already disposed of in Theorem 3.1. Let $F(+, \cdot)$ be a finite MVW system of dimension $d > 2$ over its kern K. Since K' is an associative subloop contained in $F'(\cdot)$, there exists a maximal associative subloop A containing K' and contained in $F'(\cdot)$. Lemma 3.2 now implies that $B(+, \cdot)$ is a maximal nearfield contained in $F(+, \cdot)$ where $B = A \cup \{0\}$. It is claimed that $B = F$. Suppose B is a proper subset of F. Then there is an element x in F which does not belong to B. Let B be a right vector space of dimension $t + 1$ over K. By Result 3 we can choose an element g from B such that the set $T = \{g^i \mid 0 \leqslant i \leqslant t\}$ is a basis for B over K. Let H be the group generated by $\{g, x\}$ (H exists since a Moufang loop is di-associative). Let $D = V(H; K)$. By Lemma 3.1 D is an associative subset and consequently D' which contains A properly is an associative subset contained in $F'(\cdot)$. This is a contradiction since by Result 2 A is a maximal associative set contained in $F'(\cdot)$. From this contradiction we infer the truth of the theorem.

The question of existence of infinite proper MVW systems which are not nearfields still remains unresolved. However infinite MVW systems of dimension d over their kerns for $d = 3, 4, 5, 6$ and 7 will be treated separately.

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